Passive scalar spectrum in high-Schmidt-number stationary and nonstationary turbulence

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Chasnov [Phys. Fluids **10**, 1191 (1998)] reviewed the results for passive scalar spectra in high-Schmidt-number stationary turbulence as derived by Kraichnan [J. Fluid Mech. **64**, 737 (1974)] and generalized them to simple nonstationary flows. In two-dimensional turbulence, the Kraichnan spectra are usually fitted by numerically solving the spectral equation using the derived asymptotic behavior for small and large wave numbers. In this Brief Report, we show that the Kraichnan passive scalar spectrum over the entire range of k is essentially a modified Bessel function of the second kind. We also present analytical forms of the spectra in three-dimensional nonstationary turbulence, where as shown by Chasnov, the nonstationarity can be responsible for different asymptotic behavior than the usual Kraichnan's three-dimensional stationary form. Our results considerably simplify the "fitting" of passive scalar spectra from experimental and numerical data, with the simple analytical form valid for the whole range of k, instead of just the asymptotes, which are usually valid only for a small fraction of resolved wave numbers.

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The convection of a passive scalar by a turbulent velocity field has a wide range of applications in many physical systems ranging from temperature fluctuations in grid generated turbulence and oceanic measurements, to mixing of chemicals introduced to the fluid or pollutants in the atmosphere. For extensive reviews of passive scalar mixing see, for example, Shraiman and Siggia [1], Warhaft [2], Dimotakis [3], or the book by Lesieur [4]. An important class of flows occurs when the diffusivity of the scalar D is small compared to the viscosity of the fluid ν, so the Schmidt number $\mathrm{Sc} = \nu/D > 1$. A high Schmidt-number condition means that the scalar wants to develop much smaller length scales compared to the velocity field. For stationary turbulence (when the spectrum decays self-similarly), the high Schmidt-number regime of the flow was considered by Batchelor [5] who, under the assumption that large-scale velocity fluctuations can be represented as a uniform strain for the small-scale scalar field, derived that in the viscous-convective range, the scalar spectrum is proportional to k^{-1} followed by a Gaussian falloff $\exp(-k^2)$. Kraichnan [6] generalized this idea with the inclusion of fluctuations in space and time of the strain field and showed that in the viscous-convective range the spectrum follows k^{-1} as $k \to 0$, whereas for higher k it falls off as $\exp(-k)$. Although the theory of passive scalar mixing is far from completed, Kraichnan's result seems to better fit numerical and experimental data and is generally accepted as correct. Extensive numerical analysis of passive scalar dynamics has been performed by Chasnov [7], who also presented Kraichnan's analytical results in a "more tractable" way and generalized them for simple nonstationary flows. For the three-dimensional stationary case, Kraichnan obtained the solution in simple elementary functions, but for two-dimensional flows, the spectra solutions contain confluent hypergeometric functions. Usually to fit the data, the asymptotic behavior for low and high k is considered, and then the spectral equation is integrated numerically from high k to $k=0$ [7]. In this Brief Report, we show that the solution for the passive scalar spectrum for stationary and nonstationary turbulence can be expressed as a simple modified Bessel function.

For simplicity and to facilitate a direct comparison of our results, we adopt the same nomenclature as Chasnov [7], where ν is viscosity, D is scalar diffusivity, and $\epsilon, \epsilon_\theta$ are the dissipation rates for the energy and scalar, respectively. Without sources, the passive scalar spectrum $E_\theta(k,t)$ obeys the evolution equation in k space

$$\frac{\partial}{\partial t} E_\theta(k,t) = T_\theta(k,t) - 2Dk^2 E_\theta(k,t), \tag{1}$$

where $T_\theta(k,t)$ is the scalar transfer spectrum. The scalar dissipation rate is defined as

$$\epsilon_\theta = 2D \int_0^\infty k^2 E_\theta(k,t) dk. \tag{2}$$

To eliminate the explicit time dependence, dimensional analysis shows that the spectra must be transformed using the Batchelor scaling,

$$E_\theta(k,t) = \epsilon_\theta D^{1/2} (\nu/\epsilon)^{3/4} \hat{E}_\theta(\hat{k}), \tag{3}$$

$$T_\theta(k,t) = \epsilon_\theta D^{1/2} (\nu/\epsilon)^{1/4} \hat{T}_\theta(\hat{k}), \tag{4}$$

where

$$\hat{k} = k/k_B, \quad k_B = (\epsilon/\nu D^2)^{1/4}. \tag{5}$$

The time derivative may be scaled as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \frac{d\hat{k}}{dt'} \frac{\partial}{\partial \hat{k}} = \frac{\partial}{\partial t'} - \frac{1}{4} \frac{\hat{k}}{\epsilon} \frac{d\epsilon}{dt'} \frac{\partial}{\partial \hat{k}}. \tag{6}$$

Using this scaling, and for convenience renaming $t' = t$, the passive scalar Eq. (1) is transformed to

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$$(y+3x)\hat{E}_\theta(\hat{k}) + x\hat{k}\frac{\partial}{\partial \hat{k}}\hat{E}_\theta(\hat{k}) = \hat{T}_\theta(\hat{k}) - 2\hat{k}^2\hat{E}_\theta(\hat{k}), \quad (7)$$

where $x(t)$ and $y(t)$ are nonstationary time-dependent turbulence functions and are defined as

$$x = \frac{1}{2}\frac{d}{dt}\left(\frac{\nu}{\epsilon}\right)^{1/2}, \quad y = \left(\frac{\nu}{\epsilon}\right)^{1/2}\epsilon_\theta^{-1}\frac{d\epsilon_\theta}{dt}. \quad (8)$$

For stationary turbulence, one assumes that $x=y=0$, otherwise it would contradict the assumption that the spectra are self-similar. Definition (2) is transformed to the normalization condition

$$\int_0^\infty \hat{k}^2\hat{E}_\theta(\hat{k})d\hat{k} = \frac{1}{2}. \quad (9)$$

Kraichnan proposed a form of transfer spectrum,

$$T_\theta(k,t) = -\alpha(\epsilon/\nu)^{1/2}\frac{\partial}{\partial k}\left(kE_\theta - \frac{k}{N}\frac{\partial}{\partial k}(kE_\theta)\right), \quad (10)$$

where N is the number of space dimensions, and α is the only free parameter of the model. The parameter α has been both predicted and measured by many authors, and Chasnov uses a value of $\alpha^{-1}=6.0$. Using Batchelor's scaling, Eq. (7) for stationary turbulence ($x=y=0$) shows that the normalized passive scalar spectrum $\hat{E}_\theta(\hat{k})$ is a solution of the equation ([7,6])

$$\alpha\frac{d}{d\hat{k}}\left[\hat{k}\hat{E}_\theta(\hat{k}) - \frac{\hat{k}}{N}\frac{d}{d\hat{k}}(\hat{k}\hat{E}_\theta(\hat{k}))\right] + 2\hat{k}^2\hat{E}_\theta(\hat{k}) = 0, \quad (11)$$

after using the Kraichnan form of the transfer function (10). Chasnov reduced Eq. (11) by assuming

$$\hat{E}_\theta(\hat{k}) = \alpha^{-1}\hat{k}^{-1}f(r), \quad \text{where} \quad r = (2N\alpha^{-1})^{1/2}\hat{k}, \quad (12)$$

to obtain the equation for $f(r)$ as

$$f'' - \frac{N-1}{r}f' - f = 0. \quad (13)$$

The boundary conditions from Eqs. (9) and (13) are determined to be $f(\infty)=0$ and $f(0)=1$. For $N=3$ Kraichnan [6] recognized (as did Mjolsness [8] independently) that the solution is

$$f(r) = (1+r)\exp(-r). \quad (14)$$

For $N=2$ the situation is more complicated and Chasnov solved Eq. (13) numerically using derived asymptotic solutions

$$f(r) = \left(1 + \frac{1}{2}r^2\ln r + O(r^2)\right), \quad r \to 0, \quad (15)$$

$$f(r) \sim r^{1/2}\exp(-r), \quad r \to \infty. \quad (16)$$

Chasnov starts with a sufficiently large value of r so Eq. (16) is approximately zero and numerically integrates Eq. (13) over the full range of r and adjusts the proportionality constant in Eq. (16) so that $f(0)=1$.

The solution to Eq. (13) for the case $N=2$ can be found analytically in a different way than originally considered by Kraichnan. From the theory of Bessel's functions (see, for example, Relton [9]) the solution to Eq. (13) can be cast in a general form as

$$f(r) = r[aI_1(r) + bK_1(r)], \quad (17)$$

where $I_\nu(r)$ and $K_\nu(r)$ are the modified Bessel functions of the first and second kind, respectively (of νth order, in general). Because for large r, the asymptotic behavior of the functions is

$$I_\nu(r) \to \frac{1}{\sqrt{2\pi r}}e^r, \quad K_\nu(r) \to \sqrt{\frac{\pi}{2r}}e^{-r}, \quad r \to \infty, \quad (18)$$

to prevent divergence [the boundary condition $f(\infty)=0$] requires that $a=0$. Furthermore, because for small values of r,

$$K_\nu(r) \to \frac{\Gamma(\nu)}{2}\left(\frac{2}{r}\right)^\nu, \quad r \to 0, \quad \nu \neq 0, \quad (19)$$

where $\Gamma(\nu)$ is the gamma function, the solution (17) converges to

$$f(r) \to br\frac{1}{2}\left(\frac{2}{r}\right) = b, \quad \text{as} \quad r \to 0. \quad (20)$$

Therefore the boundary condition $f(0)=1$ requires that $b=1$. The solution of Eq. (13) is therefore the simple function

$$f(r) = rK_1(r), \quad (21)$$

and Kraichnan's passive scalar spectrum for stationary turbulence in 2D is the modified Bessel function of the second kind and first order,

$$\hat{E}_\theta(\hat{k}) = 2\alpha^{-3/2}K_1(2\alpha^{-1/2}\hat{k}). \quad (22)$$

Bessel functions are well studied and most importantly they are part of common mathematical software like MATLAB or MATHEMATICA, so numerical integration is straightforward and the spectrum can be fit directly.

The solution (22) can of course be derived directly from Eq. (11), which can be rearranged in the form

$$\frac{d^2\hat{E}_\theta(\hat{k})}{d\hat{k}^2} - \frac{(N-3)}{\hat{k}}\frac{d\hat{E}_\theta(\hat{k})}{d\hat{k}} - \left(\frac{2N}{\alpha} + \frac{N-1}{\hat{k}^2}\right)\hat{E}_\theta(\hat{k}) = 0. \quad (23)$$

A differential equation with the general form

$$\frac{d^2y}{dx^2} + \frac{1-2a}{x}\frac{dy}{dx} - \left((\beta\gamma x^{\gamma-1})^2 + \frac{\nu^2\gamma^2 - a^2}{x^2}\right)y = 0, \quad (24)$$

has a solution of the form $x^aI_\nu(\beta x^\gamma)$ [with a second independent solution given by $x^aK_\nu(\beta x^\gamma)$] [9]. A comparison of Eqs. (24) and (23) gives

$$a = \frac{N}{2} - 1, \quad \gamma = 1, \quad \beta = (2N\alpha^{-1})^{1/2}, \quad \nu = \frac{N}{2}, \quad (25)$$

and after eliminating the I_ν solution because of the boundary condition $\hat{E}_\theta(\hat{k}\to\infty)=0$, we have the solution in the form

$$\hat{E}_\theta(\hat{k}) = c\hat{k}^a K_\nu(\beta\hat{k}), \tag{26}$$

where c is a constant, to be determined from the normalization condition (9). From Gradshteyn and Ryzhik [10]

$$\int_0^\infty x^\mu K_\nu(\beta x)dx = 2^{\mu-1}\beta^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right), \tag{27}$$

showing that the normalization condition (9) yields

$$c = \frac{\alpha^{-1}(2N\alpha^{-1})^{N/4}}{\Gamma(N/2)2^{(N/2-1)}}. \tag{28}$$

The solution

$$\hat{E}_\theta(\hat{k}) = c\hat{k}^{(N/2-1)}K_{N/2}[(2N\alpha^{-1})^{1/2}\hat{k}], \tag{29}$$

with normalization constant (28) is therefore the solution of the passive scalar spectrum in stationary turbulence for general dimension N. For the special case $N=2$, Eq. (28) yields $c=2\alpha^{-3/2}$, i.e., Eq. (22). Notice also, that by using Eq. (19), the solution (29) converges to $\hat{E}_\theta(\hat{k}) \to \alpha^{-1}\hat{k}^{-1}$ as $\hat{k}\to 0$ for all values of N, which is the famous $\hat{k}^{-1}$ scaling of Batchelor [5]. The solution (29) is useful for both even and odd dimensions N, because for odd dimensions, the spectrum is expressible in elementary functions. For the special case $N=3$,

$$K_{3/2}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}(1+z^{-1}), \tag{30}$$

and the three-dimensional passive scalar spectrum for stationary turbulence is therefore

$$\hat{E}_\theta(\hat{k}) = \alpha^{-1}\hat{k}^{-1}[1+(6\alpha^{-1})^{1/2}\hat{k}]\exp[-(6\alpha^{-1})^{1/2}\hat{k}], \tag{31}$$

as recognized by Kraichnan [6] and Mjolsness [8].

It is important to note that the full analytic expression for the passive scalar spectrum in stationary turbulence was obtained already by Kraichnan [6], although in a different form. Kraichnan wanted to show explicitly that for the low $\hat{k}$, the spectrum behaves like $\hat{k}^{-1}$ and that for the high $\hat{k}$, it decays as $\exp(-\hat{k})$. In our notation, Kraichnan searched for a spectrum in the form

$$\hat{E}_\theta(\hat{k}) = \alpha^{-1}\hat{k}^{-1}f(r)\exp(-r). \tag{32}$$

From Eq. (11), this yields Kummer's equation for $f(r)$,

$$rf'' - (2r+N-1)f' + (N-1)f = 0. \tag{33}$$

Kraichnan's form (32) has an advantage in that it explicitly demonstrates the behavior of the spectrum for low and high $\hat{k}$. However, the solutions of Kummer's equation are confluent hypergeometric functions which are much less "user friendly" and it is a common practice to investigate just the asymptotes of the spectrum or to use direct numerical integration as done by Chasnov. For the relevant case of two-dimensional turbulence, the form of the spectrum (22), the modified Bessel function K_1, is especially simple. For three-dimensional turbulence, the preferred expression is, of course, Eq. (31).

The solutions derived by Chasnov for nonstationary flow in the special case of x and y nonzero constants, can also be written in forms of Bessel functions. In this case, seeking a solution of the form (12) yields the evolution equation for $f(r)$ as

$$r^2f'' - \left(N-1+\frac{Nx}{\alpha}\right)rf' - \left(r^2+\frac{N(2x+y)}{\alpha}\right)f = 0. \tag{34}$$

To solve Eq. (34) Chasnov neglects the term r^2f and therefore finds a solution for $r\to 0$, with $f(r)\sim r^{-s}$, so yielding a modified form of the scalar spectrum as $\hat{E}_\theta(\hat{k})\sim\hat{k}^{-(1+s)}$ for $\hat{k}\to 0$. To find the value of s, Chasnov obtained the quadratic equation for s, and chose the root that in the limit $N\to\infty$ gives a solution that is identical to Batchelor's form, which can be derived to be

$$\hat{E}_\theta(\hat{k})^{\text{Batchelor}} \to \hat{k}^{-(1+z)}, \quad \text{as } \hat{k}\to 0, \quad z = \frac{2x+y}{\alpha+x}. \tag{35}$$

The modified form of Kraichnan's spectral index given by Chasnov [7] is then

$$s = \frac{N(\alpha+x)}{2\alpha}\left[\left(1+\frac{4\alpha z}{N(\alpha+x)}\right)^{1/2} - 1\right]. \tag{36}$$

This procedure can be followed and spectra for the full range of $\hat{k}$ can be obtained using the Relton form (24). For the nonzero, constant x,y considered by Chasnov, the spectrum $\hat{E}_\theta(\hat{k})$ obeys the equation

$$\frac{d^2\hat{E}_\theta(\hat{k})}{d\hat{k}^2} - \frac{\left(N-3+\frac{Nx}{\alpha}\right)}{\hat{k}}\frac{d\hat{E}_\theta(\hat{k})}{d\hat{k}} - \left(\frac{2N}{\alpha}+\frac{N-1+(y+3x)\frac{N}{\alpha}}{\hat{k}^2}\right)\hat{E}_\theta(\hat{k}) = 0. \tag{37}$$

The Relton form (24) therefore yields

$$a = \frac{N}{2}\left(1+\frac{x}{\alpha}\right) - 1, \quad \gamma = 1,$$

$$\beta = (2N\alpha^{-1})^{1/2}, \quad \nu = \frac{N}{2}\left(1+\frac{x}{\alpha}\right)\left(1+\frac{4\alpha z}{N(\alpha+x)}\right)^{1/2}, \tag{38}$$

with the full solution for the passive scalar spectrum given as

$$\hat{E}_\theta(\hat{k}) = c\hat{k}^a K_\nu(\beta\hat{k}). \tag{39}$$

The constant c obtained from the normalization condition using Eq. (27) is

$$c = \frac{2^{-(a+2)}\beta^{a+3}}{\Gamma\left(\frac{3+a+\nu}{2}\right)\Gamma\left(\frac{3+a-\nu}{2}\right)}. \tag{40}$$

For the special case $x=y=0$, the solution (39) is equivalent to the solution (29) as can be directly verified. The asymptotes of solution (39) are

$$\hat{E}_\theta(\hat{k}) = c\Gamma(\nu)2^{\nu-1}\beta^{-\nu}\hat{k}^{-(1+s)}, \quad \hat{k} \to 0, \tag{41}$$

$$\hat{E}_\theta(\hat{k}) = c(\pi/2\beta)^{1/2}\hat{k}^{(N-3+Nx/\alpha)/2} \exp(-\beta\hat{k}), \quad \hat{k} \to \infty, \tag{42}$$

where s is given by Eq. (36) and the normalization constants follow from Eq. (39).

We briefly illustrate the obtained analytical forms of the spectrum by performing a simple two-dimensional numerical simulation of passive scalar evolution with constant mean gradient, e.g., see Chasnov [7]. The code is an incompressible finite difference, two-step MacCormack scheme with a fast Poisson solver. The resolution was 512^2 and the time step $dt=10^{-3}$. We assumed that viscosity $\nu=10^{-3}$, diffusivity $D=10^{-4}$, yielding the Schmidt number $S_c=10$. Initially, the passive scalar in the whole 2D plane was put to zero and velocities with random phases were generated with an energy spectrum $E(k) \sim k/[1+(k/k_0)^4]$, with peak $k_0=6$ and normalized so that the total kinetic energy $E=0.5$. Values for x and y were obtained by averaging $x(t), y(t)$ between times $t=2$ and $t=10$. Without trying to find the best fit for the data, we used the value $\alpha^{-1}=6.0$, to be consistent with Chasnov. The analytical form (39) together with simulated spectrum at $t=10$ is presented in Fig. 1. A formal evaluation of Eq. (36) yielded $s=+0.43$, implying asymptotic behavior $\hat{k}^{-1.43}$ as $\hat{k} \to 0$. For comparison, we also included the stationary solution (22), representing the $\hat{k}^{-1}$ range. Solution (39) fits the obtained data quite well, although there is some overestimation in the dissipation range, a consequence of our relatively low resolution. The purpose of Fig. 1 is to compare the analytical Bessel function solutions to a simulated data set. For higher resolution and Schmidt-number results, together with averaging over different realizations of random initial conditions, see the numerical simulations of Chasnov [7].

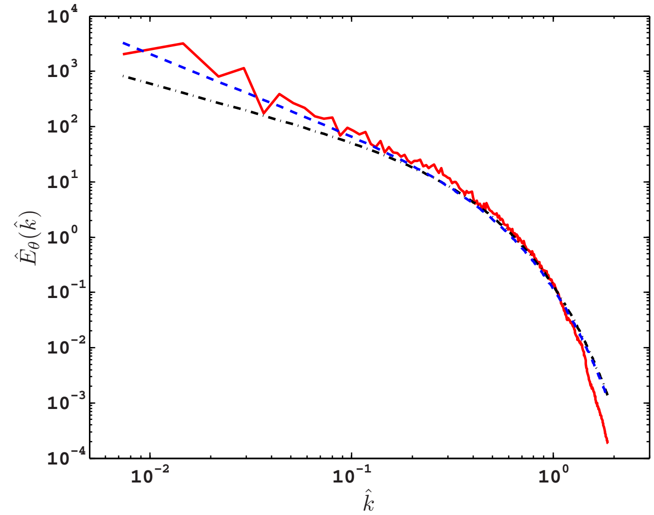

FIG. 1. (Color online) Passive scalar spectrum $\hat{E}_\theta(\hat{k})$. The solid line "—" (red) represents numerical data, while the dashed line "– –" (blue) corresponds to the nonstationary analytical form (39). For comparison we also included stationary analytical form (22) represented by the dashed-dotted line "–·–" (black). Clearly, the form (39) fits the data much better, consistent with the "nonstationary" nature of our simulations (no driving).

In summary, we reconsidered the solutions of passive scalar spectra in stationary turbulence derived by Kraichnan and in nonstationary turbulence by Chasnov and presented them in a useful form. Our solutions are appropriate for two-dimensional turbulence and also three-dimensional nonstationary turbulence, where spectra can be fit for the whole range of $\hat{k}$ as a simple modified Bessel function. It is especially useful for numerical simulations, since we can fit the full spectrum rather than fit asymptotes in either the $\hat{k}^{-1}$ or the $\hat{k}^{-(1+s)}$ ranges.

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[1] B. I. Shraiman and E. D. Siggia, Nature (London) **405**, 639 (2000).
[2] Z. Warhaft, Annu. Rev. Fluid Mech. **32**, 203 (2000).
[3] P. E. Dimotakis, Annu. Rev. Fluid Mech. **37**, 329 (2005).
[4] M. Lesieur, *Turbulence in Fluids, Stochastic and Numerical Modelling* (Kluwer Academic Publishers, Dordrecht, 1990).
[5] G. K. Batchelor, J. Fluid Mech. **5**, 113 (1959).
[6] R. H. Kraichnan, J. Fluid Mech. **64**, 737 (1974).
[7] J. R. Chasnov, Phys. Fluids **10**, 1191 (1998).
[8] R. C. Mjolsness, Phys. Fluids **18**, 1393 (1975).
[9] F. E. Relton, *Applied Bessel Functions* (Blackie & Son, Glasgow, 1946).
[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 2000).